Bayesian Reinforcement Learning: A Survey

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Bayesian RL: What

- Leverage Bayesian Information in RL problem
  - Dynamics
  - Solution space (Policy Class)
- Prior comes from System Designer
Bayesian RL: Why

- Exploration-Exploitation Trade-off
  - Posterior: current representation of world
  - Max Gain wrt Current World Belief

- Regularization
  - Prior over Value, Policy (params or class) or Model results in regularization/finite sample estimation.

- Handle Parametric Uncertainty
  - Sampling based methods, aka frequentist, are computationally intractable or very conservative.
Bayesian RL: Challenges

- Selection of the correct Representation for Prior
  - How to know ahead of time?
  - Why is that knowledge not biased?

- Decision-making process over the information state
  - Dynamic Programming over large state-action spaces was hard as it is!
  - Doing this over distributions of states (beliefs) and distributions over latent dynamics model
    Computationally much harder!
Preliminaries: POMDP

Model 4 (Partially Observable Markov Decision Process) Define a POMDP \( \mathcal{M} \) to be a tuple \( \langle \mathcal{S}, \mathcal{A}, \mathcal{O}, P, \Omega, P_0, q \rangle \) where

- \( \mathcal{S} \) is the set of states,
- \( \mathcal{A} \) is the set of actions,
- \( \mathcal{O} \) is the set of observations,
- \( P(\cdot|s, a) \in \mathcal{P}(\mathcal{S}) \) is the probability distribution over next states, conditioned on action \( a \) being taken in state \( s \),
- \( \Omega(\cdot|s, a) \in \mathcal{P}(\mathcal{O}) \) is the probability distribution over possible observations, conditioned on action \( a \) being taken to reach state \( s \) where the observation is perceived,
- \( P_0 \in \mathcal{P}(\mathcal{S}) \) is the probability distribution according to which the initial state is selected,
- \( R(s, a) \sim q(\cdot|s, a) \in \mathcal{P}(\mathbb{R}) \) is a random variable representing the reward obtained when action \( a \) is taken in state \( s \).
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Multi-armed Bandits (MAB)

Model 1 (Stochastic $K$-Armed Bandit) Define a $K$-MAB to be a tuple $\langle \mathcal{A}, \mathcal{Y}, P, r \rangle$ where

- $\mathcal{A}$ is the set of actions (arms), and $|\mathcal{A}| = K$,
- $\mathcal{Y}$ is the set of possible outcomes,
- $P(\cdot | a) \in \mathcal{P}(\mathcal{Y})$ is the outcome probability, conditioned on action $a \in \mathcal{A}$ being taken,
- $r(Y) \in \mathbb{R}$ represents the reward obtained when outcome $Y \in \mathcal{Y}$ is observed.
Bayesian MAB

- In MAB model, only unknown is outcome probability $P(*|a)$
- Use Bayesian inference to learn the outcome probability from outcomes observed
- Parameterize outcome

$$P_\theta(\cdot|a)$$

- Model our uncertainty about $\theta$
Bayesian MAB - Bernoulli with Beta Prior

\[ \theta = (\theta_1, \ldots, \theta_k) \]

\[ r(Y) = Y \]

\[ Y(a) \sim \text{Bernoulli}[\theta_a] \]

\[ \theta_a \sim \text{Beta}(\alpha_a, \beta_a) \]

\[ \theta_a | y \sim \text{Beta}(\alpha_a + y, \beta_a + y) \]
Bayesian MAB - Policy Selection

- We can represent our uncertainty about $\theta$ with posterior
- How to utilize this representation to select an adequate policy
- Want policy which minimizes regret
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UCB

- Employs optimistic policy to reduce chance of overlooking the best arm
- Starts by playing each arm once
- At time step $t$, plays arm $a$ that maximizes the following ($<r_a>$ is mean reward for arm $a$, $t_a$ is number of times arm $a$ has been played so far)

$$<r_a> + \sqrt{\frac{2 \ln t}{t_a}}$$
Bayes - UCB

- Extend UCB to Bayesian setting
- Keep posterior over expected reward of each arm
- At each step, choose the arm with the maximal posterior \((1 - \beta_t)\)-quantile, where \(\beta_t\) is of order \(1/t\)
- Using upper quantile instead of posterior mean serves the role of optimism, in the spirit of original UCB
Thompson Sampling

- $P_{post}$ is posterior over $\theta$
- Sample a parameter $\hat{\theta}$ from posterior, and select optimal action with respect to $\hat{\theta}$
- Amounts to matching action selection probability to the posterior probability of each action being optimal
Thompson Sampling

Algorithm 1 Thompson Sampling

1: $\text{TS}(P_{\text{prior}})$
   $\bullet$ $P_{\text{prior}}$ prior distribution over $\theta$
2: $P_{\text{post}} := P_{\text{prior}}$
3: for $t = 1, 2, \ldots$ do
4:   Sample $\hat{\theta}$ from $P_{\text{post}}$
5:   Play arm $a_t = \arg\max_{a \in A} \mathbb{E}_{y \sim P_{\hat{\theta}}(\cdot|a)} [r(y)]$
6:   Observe outcome $Y_t$ and update $P_{\text{post}}$
7: end for
Thompson Sampling - Beta Bernoulli

Algorithm 1 Thompson Sampling

1: \( \text{TS}(P_{\text{prior}}) \)
   - \( P_{\text{prior}} \) prior distribution over \( \theta \)

2: \( P_{\text{post}} := P_{\text{prior}} \)

3: \textbf{for} \( t = 1, 2, \ldots \) \textbf{do}

4: \quad \text{Sample } \hat{\theta} \text{ from } P_{\text{post}}

5: \quad \text{Play arm } a_t = \arg \max_{a \in A} \mathbb{E}_{y \sim P_{\hat{\theta}}(\cdot | a)} [r(y)]

6: \quad \text{Observe outcome } Y_t \text{ and update } P_{\text{post}}

7: \textbf{end for}

\[
\hat{\theta} = [\hat{\theta}_1, \ldots, \hat{\theta}_k]
\]

\[
Y(a) \sim \text{Bernoulli}[\theta_a]
\]

\[
\arg \max_{a \in A} (E_{y \sim \text{Bern}[\theta_{a_1}]} [y], E_{y \sim \text{Bern}[\theta_{a_2}]} [y], \ldots, E_{y \sim \text{Bern}[\theta_{a_k}]} [y])) =
\]

\[
\arg \max_{a \in A} (\hat{\theta}_{a_1}, \hat{\theta}_{a_2}, \ldots, \hat{\theta}_{a_k})
\]
Initialize the Beta pdf of each arm (time $t = 1$)

- Beta pdf of arm 1 ($\alpha = 1; \beta = 1$)
- Beta pdf of arm 2 ($\alpha = 1; \beta = 1$)
- Beta pdf of arm 3 ($\alpha = 1; \beta = 1$)
Sampling the Beta pdf of each arm (time \( t = 1 \))

- **Beta pdf of arm 1** (\( \alpha = 2; \beta = 1 \))
  - Sampling arm 1
  - Value: 0.3

- **Beta pdf of arm 2** (\( \alpha = 1; \beta = 1 \))
  - Sampling arm 2
  - Value: 0.25

- **Beta pdf of arm 3** (\( \alpha = 1; \beta = 1 \))
  - Sampling arm 3
  - Value: 0.43
Playing arm 3 \( (t = 1) \) and receive a reward
Update the Beta pdf of arm 3 (time $t = 2$)

Beta pdf of arm 3 ($\alpha = 1; \beta = 2$)

Arm played at time $t = 1$ is arm 3
Sampling the Beta pdf of each arm (time $t = 2$)

Beta pdf of arm 1 ($\alpha = 1; \beta = 1$)

- Sampling arm 1

Beta pdf of arm 2 ($\alpha = 1; \beta = 1$)

- Sampling arm 2

Beta pdf of arm 3 ($\alpha = 1; \beta = 2$)

- Sampling arm 3

Arm played at time $t = 1$ is arm 3

Arm played at time $t = 1$
Sampling the Beta pdf of each arm 

Beta pdf of arm 1 (α = 35; β = 4)

Beta pdf of arm 2 (α = 6; β = 5)

Beta pdf of arm 3 (α = 1; β = 3)

Arm played at time t = 48 is arm 1

Slides from https://www.youtube.com/watch?v=qhqAYfPv7mQ
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Model-based Bayesian Reinforcement Learning

- Represent our uncertainty in model parameters of MDP
- Can be thought of as a POMDP where parameters represent unobservable states
- Keep joint posterior over model parameters and physical state
- Derive optimal policy with respect to this posterior
Bayes-Adaptive MDP

- Assume discrete action/state sets
- Transition probabilities consist of multinomial distributions
- Represent our uncertainty with respect to the true parameters of the multinomial distribution using a Dirichlet distribution

\[(p_1, \ldots, p_k) \sim \text{Dir}(\phi_1, \ldots, \phi_k)\]
Bayes-Adaptive MDP

Model 6 (Bayes-Adaptive MDP) Define a Bayes-Adaptive MDP $\mathcal{M}$ to be a tuple $\langle S', A, P', P'_0, R' \rangle$ where

- $S'$ is the set of hyper-states, $S \times \Phi$,
- $A$ is the set of actions,
- $P'(. | s, \phi, a)$ is the transition function between hyper-states, conditioned on action $a$ being taken in hyper-state $(s, \phi)$,
- $P'_0 \in \mathcal{P}(S \times \Phi)$ combines the initial distribution over physical states, with the prior over transition functions $\phi_0$,
- $R'(s, \phi, a) = R(s, a)$ represents the reward obtained when action $a$ is taken in state $s$. 
BAMDP Transition Model

- The transition model of the BAMDP captures transitions between hyper-states.
- By chain rule:

\[
Pr(s', \phi' | s, a, \phi) = Pr(s' | s, a, \phi) Pr(\phi' | s, a, s', \phi)
\]
BAMDP Transition Model

- The transition model of the BAMDP captures transitions between hyper-states.
- By chain rule:

$$\Pr(s', \phi' | s, a, \phi) = \Pr(s' | s, a, \phi) \Pr(\phi' | s, a, s', \phi)$$

- First term: taking expectation over all possible transition functions

$$\Pr(s' | s, a, \phi) = \int_p \Pr(s' | s, a, \phi, p) b(p) dp = \ldots = \sum_{s'' \in S} \phi_{s, a, s''}$$
BAMDP Transition Model

\[ \Pr(s', \phi' | s, a, \phi) = \Pr(s' | s, a, \phi) \Pr(\phi' | s, a, s', \phi) \]

- Second Term: update of the posterior \( \phi \) to \( \phi' \) is deterministic

\[ \Pr(\phi' | s, a, s', \phi) \text{ is } 1 \text{ if } \phi'_{s,a,s'} = \phi_{s,a,s'} + 1, \text{ and } 0, \text{ otherwise.} \]
BAMDP Transition Model

\[ \Pr(s', \phi' | s, a, \phi) = \Pr(s' | s, a, \phi) \Pr(\phi' | s, a, s', \phi) \]

\[ = \frac{\phi_{s,a,s'}}{\sum_{s'' \in S} \phi_{s,a,s''}} \mathbb{I}(\phi'_s = \phi_{s,a,s'} + 1) \]
BAMDP - Number of States

- Initially (at $t = 0$), there are only $|S|$ states, one per real MDP state (we assume a single prior $\phi_0$ is specified).

- Assuming a fully connected state space in the underlying MDP (i.e., $P(s'|s, a) > 0, \forall s, a$), then at $t = 1$ there are already $|S| \times |S|$ states, since $\phi \rightarrow \phi'$ can increment the count of any one of its $|S|$ components. So at horizon $t$, there are $|S|^t$ reachable states in the BAMDP.

- There are clear computational challenges in computing an optimal policy over all such beliefs.
BAMDP - Value Function

- Any policy which maximizes this expression is called Bayes Optimal

\[
V^*(s, \phi) = \max_{a \in \mathcal{A}} \left[ R'(s, \phi, a) + \gamma \sum_{(s', \phi') \in S'} P'(s', \phi' | s, \phi, a) V^*(s', \phi') \right]
\]

\[
= \max_{a \in \mathcal{A}} \left[ R(s, a) + \gamma \sum_{s' \in S} \sum_{s'' \in S} \phi^a_{s, s'} \phi^a_{s', s''} V^*(s', \phi') \right].
\]
Bayes Optimal Planning

- Planning algorithms which seek a Bayes optimal policy are typically based on heuristics and/or approximations due to complexity noted above.
Planning Algorithms Seeking Bayes Optimality

- Offline value approximation
  - Compute policy apriori for any possible state and posterior
  - Compute action selection strategy to optimize expected return over hyper-states of the BAMDP
  - Intractable in most domains, these methods devise approximate algorithms which leverage structural constraints

- Online near myopic value approximation
  - In practice may be fewer than $|S|^t$ states; some trajectories will not be observed.
  - Interleave planning and execution on a step-by-step basis

- Methods with exploration bonus to achieve PAC Guarantees
  - Select actions such as to incur only a small loss compared to the optimal Bayesian policy
  - Typically employ Optimism in the Face of Uncertainty; when in doubt, an agent should act according to an optimistic model of the MDP
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Online - Bayesian Dynamic Programming

- Example of online near-myopic value approximation
- Generalization of TS
- Get estimate of Q function we would get if using transition model Pr(\theta) directly
- Convergence to optimal policy is achievable

Thompson Sampling In Bayesian RL (s, b)

```
Repeat
Sample \theta_1, ..., \theta_k \sim Pr(\theta) \quad \forall a
Q_{\theta_i}^* \leftarrow \text{solve}(MDP_{\theta_i})
\hat{Q}(s, a) \leftarrow \frac{1}{k} \sum_{i=1}^{k} Q_{\theta_i}^* (s, a)
\hat{a} \leftarrow \arg\max_a \hat{Q}(s, a)
Execute \hat{a} and receive r, s'
b(\theta) \leftarrow b(\theta)Pr(r, s'|s, a, \theta)
s \leftarrow s'
```
Online - Tree Search Approximation - Forward Search

- Select actions using a more complete characterization of the model uncertainty
- Perform forward search in the space of hyper-states
- Consider current hyper-state, build fixed-depth forward search tree containing all hyper-states reachable within some fixed planning horizon, denoted $d$
- Use dynamic programming to approximate expected return of possible actions at the root of the hyper-state
- Action with highest return is executed, and then forward search is conducted on the next hyper-state
Online - Tree Search Approximation - Forward Search

- The top node contains the initial state 1 and the prior over the model \( \Phi_0 \)

- After the first action, the agent can end up in either state 1 or state 2, and updates its posterior accordingly
Online - Tree Search Approximation - Forward Search

- The main limitation of this approach is the fact that for most domains, a full forward search (i.e., without pruning of the search tree) can only be achieved over a very short decision horizon.

- The number of nodes explored is $O(|S|^d)$.

- Also requires specifying default value function at leaf nodes (since using dynamic programming back-ups).
Online - Bayesian Sparse Sampling

- Estimates the optimal value function of a BAMDP (Equation 4.3) using Monte-Carlo sampling

- Instead of looking at all actions at each level of tree, actions are sampled according to their likelihood of being optimal, according to their Q-value distributions (as defined by Dirichlet posteriors)

- Next states are sampled according to the Dirichlet posterior on the model

- This approach requires repeatedly sampling from the posterior to find which action has the highest Q-value at each state node in the tree. This can be very time consuming, and thus, so far the approach has only been applied to small MDPs.
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Methods with exploration bonus to achieve PAC Guarantees

- Select actions such as to incur only a small loss compared to the optimal Bayesian policy

- Typically employ Optimism in the Face of Uncertainty; when in doubt, an agent should act according to an optimistic model of the MDP

- Shown to achieve bounded error in a polynomial number of steps using analysis from Probably Approximately Correct (PAC) literature
BFS3: Bayesian Forward Search Sparse Sampling

- Maintains both lower and upper bounds on the value of each state-action pair, and uses this information to direct forward rollouts in the search tree

- Consider a node $s$ in the tree, then the next action is chosen greedily with respect to the upper-bound $U(s,a)$

- The next state $s'$ is selected to be the one with the largest difference between its lower and upper bound (weighted by the number of times it was visited)
**BFS3: Bayesian Forward Search Sparse Sampling**

*Theorem* [Asmuth, 2013]: With probability at least $1 - \delta$, the expected number of sub-$\epsilon$-Bayes-optimal actions taken by BFS3 is at most $\text{BSA}(S + 1)d/\delta t$ under assumptions on the accuracy of the prior and optimism of the underlying FSSS procedure.
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Offline - Bayesian Exploration Exploitation Tradeoff in LEarning (BEETLE)

- Optimal value function for a finite-horizon POMDP can be shown to be piecewise-linear and convex; can be represented by a finite set of linear segments $\alpha_0, \ldots, \alpha_m$

- The value of a given $\alpha_i$ at a belief $b_t$ is evaluated as follows:

$$\alpha_i(b_t) = \int_S \alpha_i(s) b_t(s) ds$$

$$V_t^*(b_t) = \max_{\alpha \in \Gamma_t} \int_S \alpha(s) b_t(s)$$
Offline - Bayesian Exploration Exploitation Tradeoff in LEarning (BEETLE)

1. Hyper-states \((s, \phi)\) are sampled from random interactions with BAMDP model.
2. An equivalent continuous POMDP is solved assuming \(b = (s, \phi)\) is a belief state in that POMDP.
3. The set of \(\alpha\)-functions are constructed incrementally applying Bellman updates at the sampled hyper states using standard point-based POMDP method.
Offline - Bayesian Exploration Exploitation Tradeoff in LEarning (BEETLE)

- The constructed $\alpha$-functions can be shown to be multivariate polynomials

- The main computational challenge is that the number of terms in the polynomials increases exponentially with the planning horizon

- The key to applying it in larger domains is to leverage knowledge about the structure of the domain to limit the parameter inference to a few key parameters, or by using parameter tying (whereby a subset of parameters are constrained to have the same posterior)
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